O402. Prove that in any triangle ABC, the following inequality holds

$$\sin^2 2A + \sin^2 2B + \sin^2 2C \geq 2\sqrt{3} \sin 2A \sin 2B \sin 2C.$$

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First solution by Robert Bosch, USA

Lemma:

$$\sin^2 2A + \sin^2 2B + \sin^2 2C \geq \frac{16}{3} \sin^2 A \sin^2 B \sin^2 C.$$

Proof:

$$(\sin 2A - \sin 2B)^2 + (\sin 2B - \sin 2C)^2 + (\sin 2C - \sin 2A)^2 \geq 0,$$
$$\Leftrightarrow 3\left(\sin^2 2A + \sin^2 2B + \sin^2 2C\right) \geq (\sin 2A + \sin 2B + \sin 2C)^2,$$
$$\Leftrightarrow \sin^2 2A + \sin^2 2B + \sin^2 2C \geq \frac{16}{3} \sin^2 A \sin^2 B \sin^2 C,$$

by the well-known identity

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

Now we shall prove that

$$\frac{16}{3} \sin^2 A \sin^2 B \sin^2 C \geq 2\sqrt{3} \sin 2A \sin 2B \sin 2C.$$

By the formula for the sine of double angle this inequality is

$$\sin A \sin B \sin C \geq 3\sqrt{3} \cos A \cos B \cos C,$$

since clearly $\sin A, \sin B, \sin C$ are positive, by the same reason the left hand side is always positive, but the right one is negative for obtuse triangle, zero for the right triangle, and positive for the acute triangle. Thus our inequality becomes

$$\tan A \tan B \tan C \geq 3\sqrt{3},$$

but consider the well-known identity

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C,$$

so by AM-GM inequality we have

$$S \geq 3\sqrt[3]{S},$$

where $S = \tan A + \tan B + \tan C$, cubing is equivalent to $S \geq 3\sqrt{3}$.

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Since for nonacute triangle this inequality is obvious (because then $LHS > 0$ and $RHS \leq 0$)
So further we can assume that ABC is acute triangle, that is $A, B, C < \pi/2$. Let

$$\alpha := \pi - 2A, \beta := \pi - 2B, \gamma := \pi - 2C.$$

Then $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = \pi$ and original inequality
inequality can be equivalently rewritten as

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma \geq 2\sqrt{3} \sin \alpha \sin \beta \sin \gamma. \tag{1}$$

Let a, b, c, R and F be sidelengths circumradius and area of a triangle with angles α, β, γ.
Then multiplying inequality (1) by $4R^2$ we obtain

$$(1) \iff a^2 + b^2 + c^2 \geq \frac{\sqrt{3}abc}{R}.$$

Since $abc = 4FR$ then $\iff (1) \iff a^2 + b^2 + c^2 \geq 4\sqrt{3}F$ where latter inequality is Weitzenböck's inequality.
Just in case proof of inequality $a^2 + b^2 + c^2 \geq 4\sqrt{3}F$:
Since

$$16F^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$$

then

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}F \iff \left(a^2 + b^2 + c^2\right)^2 \geq 3\left(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\right) \iff$$

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2.$$

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